

ON THE SUBCONVEXITY PROBLEM FOR $GL(3) \times GL(2)$ L -FUNCTIONS

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ABSTRACT. Fix g a self-dual Hecke-Maass form for $SL_3(\mathbb{Z})$. Let f be a holomorphic newform of prime level q and fixed weight. Conditional on a lower bound for a short sum of squares of Fourier coefficients of f , we prove a subconvexity bound in the q aspect for $L(s, g \times f)$ at the central point.

1. INTRODUCTION

An outstanding problem in analytic number theory is to understand the size of an L -function at its central point. For an L -function $L(s)$ from the Selberg class with analytic conductor C and functional equation relating values at s and $1-s$, the Lindelöf hypothesis $L(\frac{1}{2}) \ll_{\epsilon} C^{\epsilon}$ is expected for any $\epsilon > 0$. Given an average version of the Ramanujan conjecture (which in many cases is available by the works of Iwaniec [8], Molteni [18], and Xiannan Li [15]), it only requires the functional equation to prove the so called convexity bound $L(\frac{1}{2}) \ll_{\epsilon, d} C^{\frac{1}{4}+\epsilon}$, where d denotes the degree of $L(s)$. This (or a refinement of this due to Heath-Brown [7]) is considered the trivial bound and was the best known in general until Soundararajan [20] recently proved, assuming the Ramanujan conjecture, that $L(\frac{1}{2}) \ll_{\epsilon, d} C^{\frac{1}{4}}(\log C)^{-1+\epsilon}$. The subconvexity problem is to save a power of C ; that is to prove that $L(\frac{1}{2}) \ll_{\epsilon, d} C^{\frac{1}{4}-\delta}$ for some $\delta > 0$. For L -functions of degree 1 or 2, the problem is completely solved. This involves the work of many authors, but the contribution of Friedlander and Iwaniec is particularly noteworthy. They invented the amplifier method [3, 8], which has been used to solve many cases of the subconvexity problem. For higher degree L -functions, a subconvexity bound is known only in a limited number of cases and remains a challenging and important goal.

In this paper we study certain degree 6 L -functions, the Rankin-Selberg $GL(3) \times GL(2)$ L -functions. In a recent breakthrough, Xiaoqing Li [16] proved a subconvexity bound for the L -function of a self-dual Hecke-Maass form for $SL_3(\mathbb{Z})$ twisted by a Hecke-Maass form for $SL_2(\mathbb{Z})$, or by a holomorphic Hecke cusp form for $SL_2(\mathbb{Z})$, in the eigenvalue aspect, or respectively in the weight aspect, of the $GL(2)$ form. As a corollary she derived subconvexity for a self dual $GL(3)$ form on the critical line. Blomer [2] considered this problem in the level aspect and proved subconvexity for $GL(3) \times GL(2)$ L -functions where the twist is by special Hecke-Maass forms of prime square level. For prime level however, subconvexity is still unknown and this is the problem which we consider.

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Let $H_k^*(q)$ denote the set of holomorphic cusp forms of weight k which are newforms of level q with trivial nebentypus in the sense of Atkin-Lehner Theory [1]. Fix g a self-dual Hecke-Maass form for $SL_3(\mathbb{Z})$ which is unramified at infinity. Let $L(s, g \times f)$ denote the Rankin-Selberg convolution of g with $f \in H_k^*(q)$. Kim and Shahidi [13] have shown that this is in fact an automorphic L -function. We normalize to have the central point at $s = \frac{1}{2}$. The analytic conductor in the q aspect equals q^3 , so that the convexity bound is $q^{\frac{3}{2}+\epsilon}$.

In the works of Xiaoqing Li and Blomer, a study of the first moment of the L -function at $s = \frac{1}{2}$ is enough to yield subconvexity. For example, in the weight aspect, the analytic conductor of $L(s, g \times f)$ equals k^6 so that the convexity bound is $k^{\frac{3}{2}+\epsilon}$. We further know by a result of Lapid [14] that $L(g \times f, \frac{1}{2}) \geq 0$. Hence if we had the expected (by the Lindelöf hypothesis) upper bound

$$(1.1) \quad \sum_{f \in H_k^*(q)} L(\tfrac{1}{2}, g \times f) \ll_{q, \epsilon} k^{1+\epsilon},$$

then dropping all but one term of this sum would immediately yield subconvexity. Although Li does not establish (1.1), she studies a similar first moment with an extra averaging over k . On the other hand, if we had the expected upper bound

$$(1.2) \quad \sum_{f \in H_k^*(q)} L(\tfrac{1}{2}, g \times f) \ll_{k, \epsilon} q^{1+\epsilon},$$

then dropping all but one term would not yield any useful bound in the q aspect. One could try to estimate the second moment, but this seems difficult. Thus a new ingredient is needed. We make use of an amplifier to prove

Theorem 1.1. *Fix $k > 10^6$ an even number. Let q be a prime number. Suppose that for $f_0 \in H_k^*(q)$ we have*

$$(1.3) \quad \sum_{n < L} a_{f_0}(n)^2 \gg_{\epsilon} L^{1-\epsilon}$$

for $L > q^{1/4+1/2001}$, where $a_{f_0}(n)$ is the n -th Fourier coefficient of f_0 as defined in (1.5). Then

$$(1.4) \quad L(\tfrac{1}{2}, g \times f_0) \ll q^{3/4-1/2001}.$$

One of the fundamental contributions to the subconvexity problem for degree 2 L -functions is Iwaniec's conditional proof of subconvexity for Hecke-Maass L -functions in the eigenvalue aspect [8], in which an assumption just like (1.3) is made. The assumption in the theorem is expected of all $f_0 \in H_k^*(q)$ and any $L > q^{\epsilon}$. It is known to be true for almost all $f_0 \in H_k^*(q)$ and would follow, for instance, from a strong subconvexity bound for $L(\frac{1}{2} + it, f_0 \times f_0)$ in the q aspect. It is interesting that a bound on one L -function can imply a bound on another very different one.

The exponent in our subconvexity bound and the lower bound for k are not optimal. We have concentrated on a method to break the convexity bound, leaving the task of finding the best parameters to a time when the theorem can be made unconditional.

1.2. L -functions. Every $f \in H_k^*(q)$ has a Fourier expansion of the type

$$(1.5) \quad f(z) = \sum_{n=1}^{\infty} a_f(n) n^{\frac{k-1}{2}} e(nz)$$

for $\Im z > 0$, where $e(z) = e^{2\pi iz}$, $a_f(n) \in \mathbb{R}$ and $a_f(1) = 1$. The coefficients $a_f(n)$ satisfy the multiplicative relation

$$(1.6) \quad a_f(n)a_f(m) = \sum_{\substack{d|(n,m) \\ (d,q)=1}} a_f\left(\frac{nm}{d^2}\right)$$

and Deligne's bound $a_f(n) \leq d(n) \ll n^\epsilon$. Here and throughout the paper, ϵ denotes an arbitrarily small positive constant, but not necessarily the same one from one occurrence to the next, and any implied constant may depend on ϵ . Also, q will always be a prime number. The L -function associated to f is defined as

$$(1.7) \quad L(s, f) = \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s}$$

for $\Re(s) > 1$. This satisfies the functional equation

$$(1.8) \quad q^{\frac{s}{2}} \pi^{-s} \Gamma\left(\frac{s + \frac{k-1}{2}}{2}\right) \Gamma\left(\frac{s + \frac{k+1}{2}}{2}\right) L(s, f) \\ = \epsilon_f q^{\frac{1-s}{2}} \pi^{-s} \Gamma\left(\frac{1-s + \frac{k-1}{2}}{2}\right) \Gamma\left(\frac{1-s + \frac{k+1}{2}}{2}\right) L(1-s, f),$$

where

$$(1.9) \quad \epsilon_f = -i^k a_f(q) q^{\frac{1}{2}} = \pm 1.$$

The left hand side of (1.8) analytically continues to an entire function. The facts above can be found in [9].

We fix a self-dual Hecke-Maass form g of type (ν, ν) for $SL_3(\mathbb{Z})$. We refer to [4], especially Chapter 6, and follow its notation. We write $A(n, m) = A(m, n)$ for the Fourier coefficients of g in the Fourier expansion (6.2.1) of [4], normalized so that $A(1, 1) = 1$. The L -function associated to g is defined as

$$(1.10) \quad L(s, g) = \sum_{n=1}^{\infty} \frac{A(n, 1)}{n^s}$$

for $\Re(s) > 1$. The coefficients $A(n, 1)$ are real. $L(s, g)$ is actually the symmetric-square L -function of a Hecke-Maass form for $SL_2(\mathbb{Z})$, by the work of Soudry [19]. This implies, by the work of Kim and Sarnak [12], that

$$(1.11) \quad |A(n, 1)| \ll n^{7/32+\epsilon}$$

and, by the work of Selberg, that

$$(1.12) \quad \Re(3\nu - 1) = 0.$$

We have the Hecke relation

$$(1.13) \quad A(n, m) = \sum_{d|(n,m)} \mu(d) A\left(\frac{n}{d}, 1\right) A\left(1, \frac{m}{d}\right),$$

and if $(n_1 m_1, n_2 m_2) = 1$, we have

$$(1.14) \quad A(n_1 n_2, m_1 m_2) = A(n_1, m_1) A(n_2, m_2).$$

By (1.11) and (1.13) we have

$$(1.15) \quad A(n, m) \ll (nm)^{7/32+\epsilon}.$$

By (1.11) and Rankin-Selberg theory we have (cf. [2] for a proof):

$$(1.16) \quad \sum_{n \leq x} |A(na, b)|^2 \ll x(ab)^{7/16+\epsilon}.$$

This together with (1.13) and the Cauchy-Schwarz inequality yields

$$(1.17) \quad \sum_{\substack{n \leq x \\ m \leq y}} |A(na, mb)| \ll (xy)^{1+\epsilon} (ab)^{7/32+\epsilon}.$$

The Rankin-Selberg L -function $L(s, g \times f)$ is defined as

$$(1.18) \quad L(s, g \times f) = \sum_{n, r \geq 1} \frac{A(r, n) a_f(n)}{(r^2 n)^s}$$

for $\Re(s) > 1$. It satisfies the functional equation

$$(1.19) \quad q^{\frac{3s}{2}} G(s) L(s, g \times f) = \epsilon_{g \times f} q^{\frac{3(1-s)}{2}} G(1-s) L(1-s, g \times f),$$

where $\epsilon_{g \times f} = (\epsilon_f)^3 = \epsilon_f$ and

$$(1.20) \quad G(s) = \pi^{-3s} \Gamma\left(\frac{s + \frac{k+1}{2} + 3\nu - 1}{2}\right) \Gamma\left(\frac{s + \frac{k+1}{2}}{2}\right) \Gamma\left(\frac{s + \frac{k+1}{2} + 1 - 3\nu}{2}\right) \\ \times \Gamma\left(\frac{s + \frac{k-1}{2} + 3\nu - 1}{2}\right) \Gamma\left(\frac{s + \frac{k-1}{2}}{2}\right) \Gamma\left(\frac{s + \frac{k-1}{2} + 1 - 3\nu}{2}\right).$$

The left hand side of (1.19) analytically continues to an entire function. To study $L(1/2, g \times f)$, we first express it as a weighted Dirichlet series.

Lemma 1.3. Approximate functional equation

Let

$$(1.21) \quad V(x) = \frac{1}{2\pi i} \int_{(\sigma)} x^{-s} \frac{G(\frac{1}{2} + s)}{G(\frac{1}{2})} \frac{ds}{s}$$

for $x, \sigma > 0$. We have

$$(1.22) \quad L(\tfrac{1}{2}, f \times g) = \sum_{n, r \geq 1} \frac{a_f(n) A(r, n)}{r \sqrt{n}} V\left(\frac{r^2 n}{q^{3/2+1/300}}\right) \\ + \epsilon_{g \times f} \sum_{n, r \geq 1} \frac{a_f(n) A(r, n)}{r \sqrt{n}} V\left(\frac{r^2 n}{q^{3/2-1/300}}\right).$$

For any $A > 0$ and integer $B \geq 0$ we have that

$$(1.23) \quad V^{(B)}(x) \ll_B x^B (1+x)^{-A},$$

so that the first sum in (1.22) is essentially supported on $r^2 n < q^{3/2+1/300+\epsilon}$ and the second sum is essentially supported on $r^2 n < q^{3/2-1/300+\epsilon}$.

Proof. The proof of this standard result may be found in Theorem 5.3 of [10]. \square

Note that the sums in (1.22) are of different lengths. This will result in less work with the second sum, which contains the root number $\epsilon_{g \times f}$.

1.4. Trace formula. We have Weil's estimate for the Kloosterman sum:

$$(1.24) \quad |S(n, m; c)| = \left| \sum_{h \bmod c}^* e\left(\frac{nh + m\bar{h}}{c}\right) \right| \leq (n, m, c)^{\frac{1}{2}} c^{\frac{1}{2}} d(c).$$

Here \star denotes that the summation is restricted to $(h, c) = 1$, and $\bar{h}h \equiv 1 \pmod{c}$. We will also need the following estimates for the J -Bessel function (see [6] and [21]). For $x > 0$ we have

$$(1.25) \quad J_{k-1}(x) \ll \min(x^{k-1}, x^{-1/2}).$$

For $x > 0$ and integers $B > 0$ we have

$$(1.26) \quad J_{k-1}^{(B)}(x) \ll_B x^{-B} + x^{-1/2}.$$

For any complex numbers α_f , define the weighted sum

$$(1.27) \quad \sum_{f \in H_k^*(q)}^h \alpha_f = \sum_{f \in H_k^*(q)} \frac{\alpha_f}{\zeta(2)^{-1} L(1, \text{sym}^2 f)},$$

where $L(s, \text{sym}^2 f)$ denotes the symmetric-square L -function of f . The arithmetic weights above occur naturally in the Petersson trace formula (1.29) and the following trace formula for newforms. Define

$$(1.28) \quad \Delta_{k,q}^*(n, m) = \frac{12}{q(k-1)} \sum_{f \in H_k^*(q)}^h a_f(n) a_f(m).$$

Lemma 1.5. Trace formula.

(i) We have

$$(1.29) \quad \Delta_{k,1}^*(n, m) = \delta(n, m) + 2\pi i^k \sum_{c \geq 1} \frac{S(n, m; c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{nm}}{c}\right),$$

where $\delta(n, m)$ equals 1 if $n = m$ and 0 otherwise.

(ii) Let q be a prime. If $(m, q) = 1$ and $q^2 \nmid n$ then

$$(1.30) \quad \Delta_{k,q}^*(n, m) = \delta(n, m) + 2\pi i^k \sum_{c \geq 1} \frac{S(n, m; cq)}{cq} J_{k-1}\left(\frac{4\pi\sqrt{nm}}{cq}\right) \\ - \frac{1}{q[\Gamma_0(1) : \Gamma_0((n, q))]} \sum_{i=0}^{\infty} q^{-i} \Delta_{k,1}^*(n, mq^{2i}).$$

Proof. (1.29) can be found in [9]. See Proposition 2.8 of [11] for (1.30). \square

Note that if $q|n$ then the last line of (1.30) is $\ll q^{-2+\epsilon}(nm)^\epsilon$ since $[\Gamma_0(1) : \Gamma_0(q)] > q$.

1.6. Voronoi summation. The $GL(3)$ Voronoi summation formula (1.31) was found by Miller and Schmid [17]. Goldfeld and Li [5] later gave another proof.

Lemma 1.7. $GL(3)$ Voronoi Summation. Let ψ be a smooth, compactly supported function on the positive real numbers and $(b, d) = 1$. We have

$$(1.31) \quad \sum_{n \geq 1} A(r, n) e\left(\frac{n\bar{b}}{d}\right) \psi\left(\frac{n}{N}\right) = \sum_{\pm} \frac{\pi^{\frac{3}{2}}}{2} d \sum_{\substack{n \geq 1 \\ l|dr}} \frac{A(n, l)}{nl} S\left(rb, \pm n; \frac{dr}{l}\right) \Psi^{\pm}\left(\frac{n}{d^3 N^{-1} r l^{-2}}\right),$$

where we define

(1.32)

$$\Psi^\pm(X) = X \frac{1}{2\pi i} \int_{(\sigma)} (\pi^3 X)^{-s} H^\pm(s) \tilde{\psi}(1-s) ds,$$

$$H^\pm(s) = \frac{\Gamma\left(\frac{s+3\nu-1}{2}\right) \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s+1-3\nu}{2}\right)}{\Gamma\left(\frac{1-s-1+3\nu}{2}\right) \Gamma\left(\frac{1-s}{2}\right) \Gamma\left(\frac{1-s+3\nu-1}{2}\right)} \mp i \frac{\Gamma\left(\frac{1+s+3\nu-1}{2}\right) \Gamma\left(\frac{1+s}{2}\right) \Gamma\left(\frac{1+s+1-3\nu}{2}\right)}{\Gamma\left(\frac{2-s+1-3\nu}{2}\right) \Gamma\left(\frac{2-s}{2}\right) \Gamma\left(\frac{2-s+3\nu-1}{2}\right)}$$

for $\sigma > 0$, where $\tilde{\psi}$ denotes the Mellin transform of ψ .

Writing $s = \sigma + it$, by Stirling's approximation of the gamma function we have

$$(1.33) \quad H^\pm(s) \ll_\sigma (1 + |t|)^{3\sigma}.$$

1.8. Amplifier method. Let $f_0 \in H_k^*(q)$. Define the amplifier

$$(1.34) \quad A(f) = \sum_{n < q^{1/4+1/2000}} \frac{a_{f_0}(n) a_f(n)}{\sqrt{n}}.$$

The assumption (1.3) implies that $A(f)$ is 'amplified' at $f = f_0$:

$$(1.35) \quad \sum_{n < q^{1/4+1/2000}} \frac{a_{f_0}(n)^2}{\sqrt{n}} \gg q^{1/8+1/4000-\epsilon}.$$

This can be seen by partial summation together with the following upper bound given in [18]:

$$(1.36) \quad \sum_{n < L} a_{f_0}(n)^2 \ll L^{1+\epsilon}$$

for all $L > q^\epsilon$. Theorem 1.1 will be deduced from the following.

Proposition 1.9. *We have*

$$(1.37) \quad \sum_{f \in H_k^*(q)}^h L(\tfrac{1}{2}, g \times f) A(f)^2 \ll q^{1+\epsilon}.$$

By Lapid's work, we have that $L(\tfrac{1}{2}, g \times f) \geq 0$. Now if we drop all but the term corresponding to f_0 then we have

$$(1.38) \quad \frac{1}{L(1, \text{sym}^2 f_0)} L(\tfrac{1}{2}, g \times f_0) A(f_0)^2 \ll q^{1+\epsilon}.$$

Using the trivial bound $L(1, \text{sym}^2 f_0) \ll q^\epsilon$ and the assumption (1.35), the subconvexity bound (1.4) follows.

By (1.6) we may write

$$(1.39) \quad A(f)^2 = \sum_{m < q^{1/2+1/1000}} \frac{x_m a_f(m)}{\sqrt{m}},$$

for some numbers $x_m \ll q^\epsilon$. By (1.17), Proposition 1.9 follows from

Proposition 1.10. *Let $m < q^{1/2+1/1000}$ be a natural number. We have*

$$(1.40) \quad \frac{1}{q} \sum_{f \in H_k^*(q)}^h L(\tfrac{1}{2}, g \times f) a_f(m) \ll \frac{q^\epsilon}{\sqrt{m}} \left(1 + \sum_{r < q^2} \frac{|A(r, m)|}{r}\right).$$

2. PROOF OF PROPOSITION 1.10

In this section we reduce the proof of Proposition 1.10 to two claims. By Lemma 1.3 and (1.6), we have that the left hand side of (1.40) is

$$(2.1) \quad \ll \sum_{n,r \geq 1} \frac{A(r,n)}{r\sqrt{n}} V\left(\frac{r^2 n}{q^{3/2+1/300}}\right) \Delta_{k,q}^*(n,m) \\ + q^{1/2} \sum_{n,r \geq 1} \frac{A(r,n)}{r\sqrt{n}} V\left(\frac{r^2 n}{q^{3/2-1/300}}\right) \Delta_{k,q}^*(nq,m).$$

By Lemma 1.5, the first line of (2.1) is

$$(2.2) \quad \ll \sum_{r < q^2} \frac{|A(r,m)|}{r\sqrt{m}} + \left| \sum_{n,r \geq 1} \frac{A(r,n)}{r\sqrt{n}} V\left(\frac{r^2 n}{q^{3/2+1/300}}\right) \sum_{c \geq 1} \frac{S(n,m;cq)}{cq} J_{k-1}\left(\frac{4\pi\sqrt{nm}}{cq}\right) \right| \\ + q^{-1+\epsilon} \sum_{f' \in H_k^*(1)} \left| \sum_{n,r \geq 1} \frac{A(r,n)a_{f'}(n)}{r\sqrt{n}} V\left(\frac{r^2 n}{q^{3/2+1/300}}\right) \right|.$$

Using (1.21), the last sum over n and r above can be written as an integral involving $L(s, g \times f')$. The line of integration can be moved to $-\infty$ to see that the last line of (2.2) is $\ll q^{-1+\epsilon}$. We will prove

Lemma 2.1. *Let $m < q^{1/2+1/1000}$. We have*

$$(2.3) \quad \sum_{n,r \geq 1} \frac{A(r,n)}{r\sqrt{n}} V\left(\frac{r^2 n}{q^{3/2+1/300}}\right) \sum_{c \geq 1} \frac{S(n,m;cq)}{cq} J_{k-1}\left(\frac{4\pi\sqrt{nm}}{cq}\right) \ll \frac{q^\epsilon}{\sqrt{m}}.$$

For the second line of (2.1), we first consider the contribution of the terms with $(n,q) = 1$. By Lemma 1.5 and the remark immediately following, the contribution of such terms is

$$(2.4) \quad \ll q^{1/2} \sum_{\substack{n,r \geq 1 \\ (n,q)=1}} \frac{A(r,n)}{r\sqrt{n}} V\left(\frac{r^2 n}{q^{3/2-1/300}}\right) \sum_{c \geq 1} \frac{S(nq,m;cq)}{cq} J_{k-1}\left(\frac{4\pi\sqrt{nqm}}{cq}\right) + O(q^{-1/2}).$$

In the sum above, the contribution of the terms with $c > q^{1/2}$ is $\ll q^{-100}$. Thus we may assume that $(c,q) = 1$, so that we have $S(nq,m,cq) = -S(n\bar{q},m,c)$. We may extend the sum to all natural numbers n , with an error of

$$(2.5) \quad \ll \sum_{n,r \geq 1} \frac{A(r,nq)}{r\sqrt{n}} V\left(\frac{r^2 n}{q^{1/2-1/300}}\right) \frac{1}{q} \sum_{c \geq 1} \frac{S(n,m;c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{nm}}{c}\right) \ll q^{-1/2},$$

on observing that the c -sum equals $\Delta_{k,1}^*(n,m) \ll q^\epsilon$ and using (1.17). We will prove

Lemma 2.2. *Let $m < q^{1/2+1/1000}$. We have*

$$(2.6) \quad q^{1/2} \sum_{n,r \geq 1} \frac{A(r,n)}{r\sqrt{n}} V\left(\frac{r^2 n}{q^{3/2-1/300}}\right) \sum_{c \geq 1} \frac{S(n\bar{q},m;c)}{cq} J_{k-1}\left(\frac{4\pi\sqrt{nmq}}{cq}\right) \ll q^{-1}.$$

Finally we must consider the terms of the second line of (2.1) with $q|n$. The contribution of these terms is

$$(2.7) \quad \ll \sum_{n,r \geq 1} \frac{A(r, nq)}{r\sqrt{n}} V\left(\frac{r^2 n}{q^{1/2-1/300}}\right) \Delta_{k,q}^*(nq^2, m) \ll q^{-1/2},$$

on using (1.6) and (1.9) to see that $\Delta_{k,q}^*(nq^2, m) = q^{-1} \Delta_{k,q}^*(n, m)$ and then using (1.17) to bound the sum absolutely. This completes the proof of Proposition 1.10.

2.3. Sketch. Before starting on the proofs of the lemmas presented in this section, we give a rough sketch of the argument for the main lemma, Lemma 2.1. Since $m \approx q^{1/2}$ and $r^2 n$ is essentially bounded by about $q^{3/2}$ in (2.3), the value of the J -Bessel function will be very small unless $n \approx q^{3/2}$, $r \approx q^\epsilon$ and $c \approx q^\epsilon$. Consider the range $q^{3/2} < n < 2q^{3/2}$, $r = 1$ and $c = 1$. Opening the Kloosterman sum, a part of what we must bound in (2.3) is

$$(2.8) \quad \sum_{h \bmod q}^* e(mh/q) \sum_{q^{3/2} < n < 2q^{3/2}} A(1, n) e(n\bar{h}/q).$$

The weight function V and the J -Bessel function have been ignored because they are roughly constant in this range. We apply the $GL(3)$ Voronoi summation formula to exchange the n -sum for another sum of length about $q^3/q^{3/2} = q^{3/2}$. A part of what we must bound is then

$$(2.9) \quad \sum_{h \bmod q}^* e(mh/q) \sum_{q^{3/2} < n < 2q^{3/2}} A(n, 1) S(n, h, q)$$

We have $\sum_{h \bmod q}^* e(mh/q) S(n, h, q) \approx qe(n\bar{m}/q)$. By reciprocity (the Chinese Remainder Theorem), we have $e(n\bar{m}/q) = e(n/mq)e(-n\bar{q}/m) \approx e(-n\bar{q}/m)$, since $n \approx q^{3/2} \approx mq$. Thus we must bound

$$(2.10) \quad \sum_{q^{3/2} < n < 2q^{3/2}} A(n, 1) e(-n\bar{q}/m).$$

The new modulus m of the exponential is much smaller than the original modulus q . We apply the $GL(3)$ Voronoi summation formula once again, to exchange the n -sum for another sum of length about $m^3/q^{3/2} \approx 1$. We must bound

$$(2.11) \quad \sum_{n < q^\epsilon} A(1, n) S(-n, q, m).$$

We bound this sum absolutely, using Weil's bound for the Kloosterman sum.

3. PROOF OF LEMMA 2.2

Write $S(n\bar{q}, m, c) = \sum_{h \bmod c}^* e(n\bar{q}h/c) e(mh, c)$. As noted above, by (1.25) we may assume that $c < q^{1/2}$. By (1.23), we may also assume that $r^2 < q^{3/2}$. Thus to prove Lemma 2.2, it is enough to show that

$$(3.1) \quad \sum_n A(r, n) e(n\bar{q}h/c) n^{-1/2} J_{k-1}\left(\frac{4\pi\sqrt{nmq}}{cq}\right) V\left(\frac{r^2 n}{q^{3/2-1/300}}\right) \ll q^{-2}.$$

We consider this sum in dyadic intervals. For $N > 0$, let $\omega(x)$ be a smooth function, compactly supported on $[1, 2]$ and satisfying $\omega^{(B)}(x) \ll_B 1$ and let

$$(3.2) \quad W(x) = x^{-1/2} J_{k-1}\left(\frac{4\pi\sqrt{xNm q^{-1}}}{c}\right) V\left(\frac{xr^2 N}{q^{3/2-1/300}}\right) \omega(x).$$

It is enough to show that

$$(3.3) \quad \sum_n A(r, n) e(n\overline{q}h/c) W\left(\frac{n}{N}\right) \ll q^{-3}$$

for

$$(3.4) \quad r^2 N < q^{3/2-1/300+\epsilon}$$

and

$$(3.5) \quad \frac{\sqrt{Nmq^{-1}}}{c} > q^{-1/10^5}.$$

We enforce the conditions (3.4) and (3.5) since otherwise (3.3) follows easily by (1.23) and (1.25).

Applying Lemma 1.7 to (3.3), it is enough to show that

$$(3.6) \quad \sum_{\substack{n \geq 1 \\ l|rc}} \frac{A(n, l)}{nl} S(rhq, \pm n, rc/l) \mathcal{W}\left(\frac{nNl^2}{c^3 r}\right) \ll q^{-4},$$

where

$$(3.7) \quad \mathcal{W}(X) = \int_{(\sigma)} X^{1-s} H^\pm(s) \widetilde{W}(1-s) ds,$$

for $\sigma > 0$ and

$$(3.8) \quad \widetilde{W}(1-s) = \int_1^2 x^{-s} W(x) dx.$$

By (1.26) and (3.5) we have for $B > 0$,

$$(3.9) \quad W^{(B)}(x) \ll_B \left(\frac{\sqrt{Nmq^{-1}}}{c}\right)^B q^{B/10^5}.$$

Thus, writing $s = \sigma + it$, we have by integration by parts B times,

$$(3.10) \quad \widetilde{W}(1-s) \ll_B |t|^{-B} \left(\frac{\sqrt{Nmq^{-1}}}{c}\right)^B q^{B/10^5}.$$

Using this bound with $B = \lfloor 3\sigma + 5 \rfloor$ and (1.33), we have

$$(3.11) \quad \begin{aligned} \mathcal{W}\left(\frac{nNl^2}{c^3 r}\right) &\ll_\sigma q^{10} \left(\frac{nl^2 N}{c^3 r}\right)^{-\sigma} \left(\frac{\sqrt{Nmq^{-1}}}{c}\right)^{3\sigma} q^{3\sigma/10^5} \\ &\ll q^{10} (nl^2)^{-\sigma} \left(\frac{r^2 Nm^3}{q^3}\right)^{\sigma/2} q^{3\sigma/10^5}. \end{aligned}$$

By (3.4) and the assumptions of the lemma, we have $r^2 Nm^3 \ll q^{3-1/10^4}$. Thus taking σ large enough proves (3.6).

4. PROOF OF LEMMA 2.1

We consider the n -sum in dyadic intervals. For $N_1 > 0$, let $\omega_1(x)$ be a smooth function, compactly supported on $[1, 2]$ and satisfying $\omega_1^{(B)}(x) \ll_B 1$ and let

$$(4.1) \quad W_1(x) = x^{-1/2} J_{k-1}\left(\frac{4\pi\sqrt{xN_1mq^{-2}}}{c}\right) V\left(\frac{xr^2 N_1}{q^{3/2+1/300}}\right) \omega_1(x).$$

It is enough to show that

$$(4.2) \quad \sum_{n \geq 1} A(r, n) S(n, m; cq) W_1\left(\frac{n}{N_1}\right) \ll \frac{q^{1+\epsilon} \sqrt{N_1}}{\sqrt{m}}$$

for

$$(4.3) \quad \begin{aligned} q^{3/2-1/990} &< N_1 < q^{3/2+1/300+\epsilon}, \\ q^{1/2-1/290} &< m < q^{1/2+1/1000}, \\ r &< q^{1/450}, \\ c &< q^{1/450}. \end{aligned}$$

We may assume the conditions above, since otherwise (4.2) follows easily by (1.23) and (1.25). Thus it is enough to prove that

$$(4.4) \quad \sum_{n \geq 1} A(r, n) S(n, m; cq) W_1\left(\frac{n}{N_1}\right) \ll q^{3/2-1/990}.$$

We apply Lemma 1.7 to the left hand side of (4.4) after writing $S(n, m; cq) = \sum_{h \bmod cq}^* e((n\bar{h} + mh)/cq)$. We need to show that

$$(4.5) \quad cq \sum_{h \bmod cq}^* e(mh/cq) \sum_{\substack{n \geq 1 \\ l|cqr}} \frac{A(n, l)}{nl} S(rh, n; qcr/l) W_1\left(\frac{nN_1 l^2}{c^3 q^3 r}\right) \ll q^{3/2-1/990},$$

where

$$(4.6) \quad \mathcal{W}_1(X) = X \int_{(\sigma)} (\pi^3 X)^{-s} H^\pm(s) \widetilde{W}_1(1-s) ds$$

for $\sigma > 0$. Note that $W_1^{(B)}(x) \ll_B q^{B/450}$ so that by integrating by parts B times we have for $s = \sigma + it$,

$$(4.7) \quad \widetilde{W}_1(1-s) = \int_1^2 x^{-s} W_1(x) dx \ll_B |t|^{-B} q^{B/450}.$$

We can use this bound together with (1.33) to estimate $\mathcal{W}_1(X)$. If $X > q^{1/150+\epsilon}$, we can take σ in (4.6) to be very large and $B = \lfloor 3\sigma + 5 \rfloor$ in (4.7) to see that $\mathcal{W}_1(X) \ll X^{-2} q^{-100}$. If $X \leq q^{1/150+\epsilon}$, we take $\sigma = \epsilon$ in (4.6) and $B = 2$ in (4.7) to see that

$$(4.8) \quad \mathcal{W}_1(X) \ll q^{1/200} X.$$

So the n -sum in (4.5) is essentially supported on $n < \frac{q^{3+1/150+\epsilon} c^3 r}{N_1 l^2} < q^{3/2+1/60}$.

The contribution to (4.5) by the terms with $q|l$ is negligible since if $q|l$ then $\frac{nN_1 l^2}{c^3 q^3 r} > q^{1/150+\epsilon}$ and we have just seen that then $\mathcal{W}_1\left(\frac{nN_1 l^2}{c^3 q^3 r}\right) \ll q^{-100}$. Henceforth fix $l|cr$, so that $l < q^{1/225}$.

We open the Kloosterman sum: $S(rh, n; qcr/l) = \sum_{u \bmod qcr/l}^* e((rhu + n\bar{u})l/qcr)$. For (4.5), it is enough to show that

$$(4.9) \quad \sum_{u \bmod qcr/l}^* \sum_{n \geq 1} \frac{A(n, l) e(n\bar{u}l/qcr)}{n} W_1\left(\frac{nN_1 l^2}{c^3 q^3 r}\right) \sum_{h \bmod cq}^* e(h(ul+m)/cq) \ll q^{1/2-1/300}.$$

The innermost sum above, a Ramanujan sum, equals

$$(4.10) \quad \left(\sum_{h \bmod q}^* e(h(ul+m)/q) \right) \left(\sum_{h \bmod c}^* e(h(ul+m)/c) \right),$$

since $(c, q) = 1$. Note that $\sum_{h \bmod q}^* e(h(ul+m)/q)$ equals -1 or $q-1$ according as $ul \not\equiv m \bmod q$ or $ul \equiv m \bmod q$ respectively. So the left hand side of (4.9) equals

$$(4.11) \quad - \sum_{n \geq 1} \frac{A(n, l)}{n} \mathcal{W}_1 \left(\frac{nNl^2}{c^3 q^3 r} \right) \sum_{h \bmod c}^* e(hm/c) S(n, qhr; qcr/l) \\ + q \sum_{n \geq 1} \frac{A(n, l)}{n} \mathcal{W}_1 \left(\frac{nNl^2}{c^3 q^3 r} \right) \sum_{h \bmod c}^* e(hm/c) \sum_{\substack{u \bmod qcr/l \\ u \equiv m\bar{l} \bmod q}}^* e \left(\frac{uqhr + \bar{u}n}{qcr/l} \right).$$

Since $(cr/l, q) = 1$, we have $S(n, qhr; qcr/l) = S(n\bar{q}, hr; cr/l) S(0, n; q)$. This product of a Kloosterman sum and a Ramanujan sum is $\ll q^{1+1/225}$ if $q|n$ and $\ll q^{1/225}$ otherwise. In any case, using (1.17), the first line of (4.11) satisfies the bound required in (4.9). Now consider the second line. By the Chinese Remainder Theorem, for $(u, qcr/l) = 1$ and $u \equiv m\bar{l} \bmod q$, we can write $u = m\bar{c}\bar{r}(cr/l) + vq$, where $cr\bar{c}\bar{r} \equiv 1 \bmod q$ and $(v, cr/l) = 1$. We have

$$(4.12) \quad e \left(\frac{uqhr}{qcr/l} \right) = e \left(\frac{vhrq}{cr/l} \right), \\ e \left(\frac{n\bar{u}}{qcr/l} \right) = e \left(\frac{n\bar{u}cr/l}{q} \right) e \left(\frac{n\bar{u}\bar{q}}{cr/l} \right) = e \left(\frac{nl^2 \bar{m} \bar{c} \bar{r}}{q} \right) e \left(\frac{n\bar{v} \bar{q}^2}{cr/l} \right).$$

Thus (4.9) is reduced to showing

$$(4.13) \quad \sum_{n \geq 1} \frac{A(n, l)}{n} e \left(\frac{nl^2 \bar{m} \bar{c} \bar{r}}{q} \right) \mathcal{W}_1 \left(\frac{nNl^2}{c^3 q^3 r} \right) \sum_{h \bmod c}^* e \left(\frac{hm}{c} \right) S(n\bar{q}^2, hrq; cr/l) \\ \ll q^{-1/2-1/300}.$$

Now comes a crucial step. By the Chinese Remainder Theorem we have

$$(4.14) \quad e \left(\frac{nl^2 \bar{m} \bar{c} \bar{r}}{q} \right) = e \left(\frac{nl^2}{mcrq} \right) e \left(\frac{-nl^2 \bar{q}}{mcr} \right).$$

For $n < q^{3/2+1/60}$, the exponential factor $e \left(\frac{nl^2}{mcrq} \right)$ has amplitude at most $q^{1/30}$ and will be absorbed into the weight function. Let $N_2 < q^{3/2+1/60+\epsilon}$, let $\omega_2(x)$ be a smooth function, compactly supported on $[1, 2]$ and satisfying $\omega_2^{(B)}(x) \ll_B 1$ and let

$$(4.15) \quad W_2(x) = x^{-1} e \left(\frac{xN_2 l^2}{mcrq} \right) \mathcal{W}_1 \left(\frac{xN_2 N_1 l^2}{c^3 q^3 r} \right) \omega_2(x).$$

It is enough to prove that

$$(4.16) \quad \sum_{n \geq 1} A(n, l) e \left(\frac{-nl^2 \bar{q}}{mcr} \right) W_2 \left(\frac{n}{N_2} \right) \sum_{h \bmod c}^* e \left(\frac{hm}{c} \right) S(n\bar{q}^2, hrq; cr/l) \\ \ll N_2 q^{-1/2-1/300}.$$

Combining the exponential factors, it is enough to prove that

$$(4.17) \quad \sum_{n \geq 1} A(n, l) e(n\bar{b}/d) W_2\left(\frac{n}{N_2}\right) \ll N_2 q^{-1/2-1/100}.$$

for $d < q^{1/2+1/150}$ and $(b, d) = 1$. We use the Voronoi summation formula again. Applying Lemma 1.7 to the left hand side of (4.17), it suffices to show that

$$(4.18) \quad d \sum_{\substack{n \geq 1 \\ \ell | dl}} \frac{A(\ell, n)}{n\ell} S(lb, n; dl/\ell) \mathcal{W}_2\left(\frac{nN_2 l^2}{d^3 r}\right) \ll N_2 q^{-1/2-1/100}.$$

where

$$(4.19) \quad \mathcal{W}_2(X) = X \int_{(\sigma)} (\pi^3 X)^{-s} H^\pm(s) \widetilde{W}_2(1-s) ds$$

for $\sigma > 0$. We need to estimate $\mathcal{W}_2(X)$. To this end we first note, using (4.8), that

$$(4.20) \quad W_2^{(B)}(x) \ll_B q^{B/30} q^{1/200} \frac{N_2 N_1 l^2}{c^3 q^3 r} \ll q^{(B+1)/30} q^{-3/2} N_2.$$

Integrating by parts B times, we have for $s = \sigma + it$ the bound

$$(4.21) \quad \widetilde{W}_2(1-s) = \int_1^2 x^{-s} W_2(x) dx \ll_B |t|^{-B} q^{(B+1)/30} q^{-3/2} N_2.$$

Now we can estimate $\mathcal{W}_2(X)$. We take $\sigma = 1$ in (4.19) and $B = 5$ above. By (1.33) and (4.21) we see that

$$(4.22) \quad \mathcal{W}_2(X) \ll N_2 q^{-3/2+1/5}.$$

Taking $\sigma = 2$, we also observe that the sum in (4.18) can be restricted to $n < q^{100}$, say, with negligible error.

Using (4.22), to prove (4.18) it is enough to show that

$$(4.23) \quad \sum_{\substack{n < q^{100} \\ \ell | dl}} \frac{|A(\ell, n)|}{n\ell} |S(lb, n; dl/\ell)| \ll q^{1/2-13/60}.$$

By (1.24), we have $|S(lb, n; dl/\ell)| \ll (lb, n, dl/\ell)^{1/2} (dl)^{1/2+\epsilon} \ll q^{1/4+1/100}$, since $(b, d) = 1$. This establishes (4.23), using (1.17).

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